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APPLICATION OF A VARIATIONAL METHOD TO FLOW OVER A FLAT PLATE IN THE ENTRANCE REGION WITH VARIABLE PHYSICAL PROPERTIES

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Abstract—A variational method has been used to solve the flow over a flat plate in the entrance region at constant wall temperature. The physical properties, i.e. thermal conductivity and viscosity, were assumed to be linear functions of temperature in the study. Two coupled equations were derived from the variational formulation and then solved by the analog/hybrid computer. Consequently, momentum boundary layer thickness, thermal boundary layer thickness local Nusselt number and local friction factor were found for the flow. For the constant properties case a comparison was made between the exact solution, and results obtained using the solution approach suggested in this paper.

NOMENCLATURE

A,	viscosity coefficient;
Amp,	amplifier;
a,	analog input signal;
B,	conductivity coefficient;
b,	analog output signal;
C,	comparator;
C_p,	specific heat at constant pressure;
D/A,	direct/analog (switch);
e_0,	output analog signal;
e_1, e_2,	input analog signal;
E,	local potential;
f,	local friction factor;
H.G.,	high gain;
h,	heat transfer coefficient;
IC,	initial condition;
K,	amplification;
k,	thermal conductivity;
L,	logic signal;
l,	characteristic length of the flate plate;
Nu_x,	local Nusselt number;
P,	potentiometers, set by servomotor;
p,	pressure;
Pr,	Prandtl number;
q,	heat flux;
Re,	Reynolds number;
s,	surface;
S1, S2,	switch 1, switch 2;
T,	temperature;
t,	time;
u,	velocity in the x-direction;
v,	velocity in the y-direction;
V,	volume;
x_0,	unheated starting length;
x, y,	Cartesian coordinates;
Y,	(Δ_t/Δ) ratio between thermal and momentum boundary layer thickness.

Greek symbols

- α, thermal diffusivity;
- Δ, momentum boundary layer thickness;
- Δ_t, thermal boundary layer thickness;
- δ, variation notation;
- θ, dimensionless temperature variable;
- μ, dynamic viscosity;
- ν, kinematic viscosity;
- ρ, density.

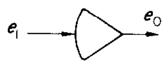
Subscripts

- i, j, tensorial indices;
- $x, (y)$, derivative with respect to x (y);
- w, wall property;
- ∞, free stream property.

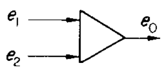
Superscripts

- $*$, dimensionless quantity;
- o, stationary state.

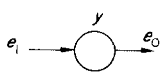
PROGRAMMER SYMBOLS

1. 

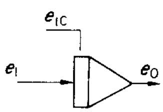
High gain dc amplifier
$e_0 = -Ke_1$ (K large, normally greater than 10^8)

2. 

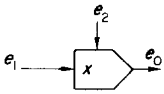
Summer–inverter
$e_0 = -(e_1 + e_2)$

3. 

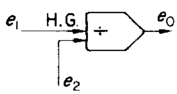
Grounded potentiometer
$e_0 = y\,e_1$
$0 \leqslant y \leqslant 1$

4. 

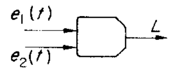
Integrator
$e_0 = -\int e_1\,dt - e_{IC}$

5. 

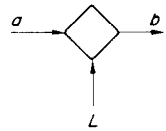
Multiplier
$e_0 = -e_1 e_2$

6. 

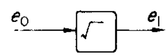
Divider
$e_0 = -e_1/e_2$

7. 

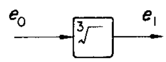
Comparator
$e_1(t) + e_2(t) > 0$,
L is TRUE
$e_1(t) + e_2(t) < 0$,
L is FALSE

8. 

D/A SWITCH
If L is TRUE a and b are connected
If L is FALSE b is grounded

9. 

$e_0 = \sqrt{e_1}$

10. 

$e_0 = \sqrt[3]{e_1}$

INTRODUCTION

An important formulation of the variational principle in thermoscience was derived by Glansdorff *et al.* [1] in 1962 based on minimum entropy production. Its application is discussed in the literature [2, 3]. However, the formulation is only applicable to the rather narrow class of systems for which

(1) the phenomenological coefficients are constant or expressed in specific forms
(2) the Onsager reciprocal relations are valid
(3) the convective terms are negligible.

Later, Glansdorff and Prigogine [4] removed the above restrictions by modifying the formulation using the concept of local potential (generalized entropy production). The application of this formulation is also discussed in the literature [5–7].

The purpose of this study is to apply the variational method based on the local potential theory to solve the problem of flow over a flat plate in the entrance region at constant wall temperature with variable physical properties.

This involves the formulation of two coupled momentum and energy boundary equations. The solution of these equations was obtained via an analog/hybrid computer. The quantities to be determined include momentum and thermal boundary layer thickness, local Nusselt number and local friction factor. In certain cases a comparison will be made between the results from this work and those available from the literature, i.e. constant properties case.

The combination of the variational method and an analog/hybrid computer in solving this class of engineering problem is to the authors' knowledge, a new approach. As a result of this study some degree of confidence in this combined solution approach is established. It is hoped that the approach can be effectively utilized in obtaining solution to other problems in this area.

VARIATIONAL FORMULATION OF THE PROBLEM

The conservation equations of mass, momentum and energy for a two-dimensional, incompressible boundary layer flow [8] can be expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{3}$$

Although the problem under study is a steady state case, one must nevertheless retain the time-dependent character of the equations when forming the local potential.

The pressure gradient is assumed to be zero for flow over a flat plate. In addition, the heat dissipation is neglected in this study [9]. Thus equations (2) and (3) become

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{4}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right). \tag{5}$$

Here μ and k are functions of T(temperature).

The closed form solution of simultaneous equations (1), (4) and (5) is, in general, very difficult to obtain even for the simplest geometry because of the non-linearities involved. The variational technique can, however, be used to transform these equations into a more tractable form.

In order to construct a local potential for the problem for use in the variational method, a technique used by Glansdorff and Prigogine [4] is followed. Upon multiplying equation (1) by $-(\rho/2)$ $(\partial v^2/\partial t)$, equation (4) by $\partial u/\partial t$ and equation (5) by $\partial T/\partial t$, summing the results and rearranging the terms, one obtains

$$\begin{aligned}\psi = -\rho\left(\frac{\partial u}{\partial t}\right)^2 - \rho\, C_p\left(\frac{\partial T}{\partial t}\right)^2 &= \rho u\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} \\ &+ \rho v\frac{\partial u}{\partial t}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t}\left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right] \\ &- \frac{\rho}{2}\frac{\partial v^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho\, C_p u\frac{\partial T}{\partial x}\frac{\partial T}{\partial t} \\ &+ \rho\, C_p v\frac{\partial T}{\partial y}\frac{\partial T}{\partial t} - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial t}\frac{\partial T}{\partial y}\right) \\ &+ \frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right)^2 \leqslant 0.\end{aligned} \tag{6}$$

Equations (1), (4) and (5) describe a two-dimensional flow. The arguments of the variational techniques require the specification of the function ϕ as follows:

$$\phi = \iint_s \psi \, \mathrm{d}x\mathrm{d}y \leqslant 0 \tag{7}$$

where s is an area of interest in the x–y plane which is bounded by curve c (the curve which encloses s).

The integrand ψ can be rearranged in the form

$$\psi = \frac{\partial}{\partial x}\left(\rho u^2 \frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(\rho uv \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\frac{\partial u}{\partial t}\right)$$
$$-\rho u \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \rho u^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$$
$$-\rho uv \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) + \frac{\mu}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\rho}{2}\frac{\partial v^2}{\partial t}$$
$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho C_p u \frac{\partial T}{\partial x}\frac{\partial T}{\partial t}$$
$$\rho C_p v \frac{\partial T}{\partial y}\frac{\partial T}{\partial t} - \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial t}\frac{\partial T}{\partial y}\right)$$
$$+ \frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right)^2 \leqslant 0. \quad (8)$$

Combining equations (7) and (8) and using Gauss' theorem, one obtains

$$\phi = \iint_s \left[-\rho u^2 \frac{\partial}{\partial t}\frac{\partial u}{\partial x} - \rho uv \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\right.$$
$$+ \frac{\mu}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\rho}{2}\frac{\partial u^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
$$-\frac{\rho}{2}\frac{\partial v^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho C_p u \frac{\partial T}{\partial x}\frac{\partial T}{\partial t}$$
$$\left. + \rho C_p v \frac{\partial T}{\partial y}\frac{\partial T}{\partial t} + \frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right)^2\right] \mathrm{d}x\mathrm{d}y$$
$$+ \int_c \left(\rho u^2 \frac{\partial u}{\partial t}\mathrm{d}y - \rho uv \frac{\partial u}{\partial t}\mathrm{d}x + \mu \frac{\partial u}{\partial y}\frac{\partial u}{\partial t}\mathrm{d}x\right.$$
$$\left. + k \frac{\partial T}{\partial y}\frac{\partial T}{\partial t}\mathrm{d}x\right) \leqslant 0. \quad (9)$$

Near the stationary state, ϕ becomes

$$\phi = \frac{\partial}{\partial t}\iint_S \left[-\rho u^{\circ 2}\frac{\partial u}{\partial x} - \rho u^\circ v^\circ \frac{\partial u}{\partial y} + \frac{\mu^\circ}{2}\left(\frac{\partial u}{\partial y}\right)^2\right.$$
$$-\frac{\rho u^2}{2}\left(\frac{\partial u^\circ}{\partial x} + \frac{\partial v^\circ}{\partial y}\right) - \frac{\rho v^2}{2}\left(\frac{\partial u^\circ}{\partial x} + \frac{\partial v^\circ}{\partial y}\right)$$
$$+ \rho C_p u^\circ T \frac{\partial T^\circ}{\partial x} + \rho C_p v^\circ T \frac{\partial T^\circ}{\partial y}$$
$$\left. + \frac{k^\circ}{2}\left(\frac{\partial T}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y + \frac{\partial}{\partial t}\int_c \left(\rho u^{\circ 2}\, u\mathrm{d}y\right.$$
$$\left. - \rho u^\circ v^\circ u\mathrm{d}x + \mu^\circ \frac{\partial u^\circ}{\partial y} u\mathrm{d}x + k^\circ \frac{\partial T^\circ}{\partial y} T\mathrm{d}x\right)$$
$$\leqslant 0. \quad 10)$$

Therefore, the local potential is

$$E = \iint_s \left[-\rho u^{\circ 2}\frac{\partial u}{\partial x} - \rho u^\circ v^\circ \frac{\partial u}{\partial y} + \frac{\mu^\circ}{2}\left(\frac{\partial u}{\partial y}\right)^2\right.$$
$$-\rho\left(\frac{u^2}{2} + \frac{v^2}{2}\right)\left(\frac{\partial u^\circ}{\partial x} + \frac{\partial v^\circ}{\partial y}\right) + \rho C_p u^\circ T \frac{\partial T^\circ}{\partial x}$$
$$\left. + \rho C_p v^\circ T \frac{\partial T^\circ}{\partial y} + \frac{k^\circ}{2}\left(\frac{\partial T}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y$$
$$+ \int_c \left(\rho u^{\circ 2} u\mathrm{d}y - \rho u^\circ v^\circ u\mathrm{d}x + \mu^\circ \frac{\partial u^\circ}{\partial y} u\mathrm{d}x\right.$$
$$\left. + k^\circ \frac{\partial T^\circ}{\partial y} T\mathrm{d}x\right) \quad (11)$$

with the subsidiary conditions

$$u^\circ = u$$
$$v^\circ = v$$
$$T^\circ = T.$$

The line integral portion of equation (11) can be simplified by using the boundary conditions [7]. In this study the area of interest is a rectangle bounded by the lines $x = 0$, $x = l$, $y = 0$ and $y = \Delta$. Note that the boundary conditions for the problem are

$$u = 0 \quad \text{at} \quad y = 0$$
$$v = 0 \quad \text{at} \quad y = 0$$
$$u = u_\infty \quad \text{at} \quad y = \Delta$$
$$u = u_\infty \quad \text{at} \quad x = 0$$

and

$$T = T_w \quad \text{at} \quad y = 0$$
$$T = T_\infty \quad \text{at} \quad y = \Delta_t$$
$$T = T_\infty \quad \text{at} \quad x = 0.$$

Therefore, the contribution from the line integral is

$$E_{\text{line}} = \int_c \left(\rho u^{\circ 2} u \mathrm{d}y + \rho u^\circ v^\circ u \mathrm{d}x - \mu^\circ \frac{\partial u^\circ}{\partial y} u \mathrm{d}x - k^\circ \frac{\partial T^\circ}{\partial y} T \mathrm{d}x\right) = \int_0^\Delta (\rho u^{\circ 2} u|_{x=l} - \rho u^{\circ 2} u|_{x=0}) \,\mathrm{d}y + \int_0^l \left[\rho u^\circ v^\circ u|_{y=\Delta} - \rho u^\circ v^\circ u|_{y=0} - \mu^\circ \frac{\partial u^\circ}{\partial y} u|_{y=\Delta} + \mu^\circ \frac{\partial u^\circ}{\partial y} u|_{y=0} - k^\circ \frac{\partial T^\circ}{\partial y} T|_{y=\Delta_t} + k^\circ \frac{\partial T^\circ}{\partial y} T|_{y=0}\right]^{\mathrm{d}x}.$$

Imposing the boundary conditions, one obtains

$$E_{\text{line}} = \int_0^\Delta (\rho u^{\circ 2} u|_{x=l} - \rho u_\infty^3)\,\mathrm{d}y + \int_0^l \rho u_\infty^2 v^\circ|_{y=\Delta}\,\mathrm{d}x. \qquad (12)$$

By using equations (1) and (12), equation (11) can be further reduced to

$$E = \iint \left[-\rho u^{\circ 2} \frac{\partial u}{\partial x} - \rho u^\circ v^\circ \frac{\partial u}{\partial y} + \frac{\mu^\circ}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \rho C_p u^\circ T \frac{\partial T^\circ}{\partial x} + \rho C_p v^\circ T \frac{\partial T^\circ}{\partial y} + \frac{k^\circ}{2}\left(\frac{\partial T}{\partial y}\right)^2\right] \mathrm{d}x\mathrm{d}y + \int_0^\Delta (\rho u^{\circ 2} u|_{x=l} - \rho u_\infty^3)\mathrm{d}y - \int_0^l \rho u_\infty^2 v^\circ|_{y=\Delta}\mathrm{d}x \qquad (13)$$

or

$$E = \iint F(x, y, u, u_x, u_y, T, T_x, T_y)\,\mathrm{d}x\mathrm{d}y + \int_0^\Delta (\rho u^{\circ 2} u|_{x=l} - \rho u_\infty^3)\,\mathrm{d}y + \int_0^l \rho u_\infty^2 v^\circ|_{y=\Delta}\,\mathrm{d}x \qquad (13a)$$

where

$$F(x, y, u, u_x, u_y, T, T_x, T_y) = -\rho u^{\circ 2}\frac{\partial u}{\partial x} - \rho u^\circ v^\circ \frac{\partial u}{\partial y} + \frac{\mu^\circ}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \rho C_p u^\circ T \frac{\partial T^\circ}{\partial x} + \rho C_p v^\circ T \frac{\partial T^\circ}{\partial y} + \frac{k^\circ}{2}\left(\frac{\partial T}{\partial y}\right)^2. \qquad (13b)$$

Again it can be seen that the local potential as defined for two-dimensional boundary layer is composed of two parts. One part is an area integral and the other is a line integral. The line integral enters because the velocity u is not specified at $x = l$. Hence the variation in velocity does not vanish at $x = l$ as it does for $x = 0$, $y = 0$ and $y = \Delta$. This problem corresponds to the one-free end boundary condition in the calculus of variations [10, 11]. Therefore, the boundary condition at $x = l$ must be a natural boundary condition.

In order to prove that equation (13) is the local potential of the problem the following operations must be performed. Taking variation of local potential E (equation (13a)) with respect to u and T, one obtains

$$\left(\frac{\delta E}{\delta u}\right)_{u^\circ} = 0 \qquad (14)$$

$$\left(\frac{\delta E}{\delta T}\right)_{T^\circ} = 0. \qquad (15)$$

Equation (14) can be written as

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0. \qquad (16)$$

Substituting equation (13b) into equation (16), it follows that

$$\frac{\partial}{\partial x}(-\rho u^{\circ 2}) - \frac{\partial}{\partial y}(-\rho u^\circ v^\circ) - \frac{\partial}{\partial y}\left(\mu^\circ \frac{\partial u}{\partial y}\right) = 0. \qquad (17)$$

Using the subsidiary conditions

$$u^\circ = u$$
$$v^\circ = v$$

Equation (17) becomes

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

or

$$2\rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0.$$

Rearranging the terms, the above equation gives

$$\rho u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

or

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right).$$

This is the momentum boundary layer equation in the x-direction for constant pressure. Similarly, from equation (15) it can be shown that

$$\rho C_p u \frac{\partial T}{\partial x} + \rho C_p v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right).$$

This is the energy boundary layer equation. Thus it has been shown that equation (13) is the local potential of the problem.

In order to proceed, the following velocity and temperature profiles are assumed:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \quad 0 \leqslant y \leqslant \Delta \qquad (18)$$

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\left(\frac{y}{\Delta_t}\right) - \frac{1}{2}\left(\frac{y}{\Delta_t}\right)^3 \quad 0 \leqslant y \leqslant \Delta_t\,. \qquad (19)$$

These satisfy the boundary conditions

$$\text{at } y = 0;\ u = 0,\ T = T_w$$

$$\text{at } y = \Delta;\ u = u_\infty,\ \frac{\partial u}{\partial y} = 0$$

$$\text{at } y = \Delta_t;\ T = T_\infty,\ \frac{\partial T}{\partial y} = 0.$$

In this study, only the case where $\Delta > \Delta_t$ is considered, where Δ, Δ_t are functions of x to be determined.*

* The approach here is similar to that of the Kantorovich Method.

For simplicity, the viscosity and thermal conductivity are chosen as linear functions of temperature i.e.

$$\frac{\mu}{\mu_\infty} = 1 + A\theta \qquad (20)$$

$$\frac{k}{k_\infty} = 1 + B\theta \qquad (21)$$

where

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}.$$

In the above expressions, a positive A and a negative B indicate cooling of the fluid, while a negative A and a positive B indicate heating of the fluid.

These expressions for velocity, temperature, viscosity and conductivity are substituted into equation (13). By imposing certain variational arguments, the following two coupled equations are obtained. (The rather tedious calculation involved in making this step are given in the Appendix).

$$2Y^2\Delta^{*2}Y' = a_5 - Y^3\Delta^*\Delta^{*\prime} \qquad (22)$$

$$-a_1\Delta^*\Delta^{*\prime} = a_2 - a_3 Y + a_4 Y^3 \qquad (23)$$

where

$$a_1 = \frac{21}{320}Re_\infty$$

$$a_2 = \frac{3}{5}$$

$$a_3 = \frac{9}{4}A\frac{3}{8}$$

$$a_4 = \frac{128}{9Pr_\infty Re_\infty}\left(\frac{3}{5} + \frac{177}{320}B\right).$$

Here $Y = \Delta_t/\Delta$ and the dimensionless quantities x^*, Δ^* and Δ_t^* are defined as

$$x^* = x/l$$

$$\Delta^* = \Delta/l$$

$$\Delta_t^* = \Delta_t/l$$

The analog/hybrid computer solution of equations (22) and (23) is discussed in the next section.

ANALOG/HYBRID COMPUTER SOLUTION TO THE PROBLEM*

Note first that the solution of the problem of simultaneous development of momentum and thermal boundary layer is intractable, because the leading edge of the flat plate is a singular point, i.e. at this point $Y = 0/0$ and hence is undefined. One way of circumventing this difficulty is to assume an unheated starting length. This length is denoted by x_0 as shown in Fig. 1 and its dimensionless counterpart is $x_0^* = x_0/l$.

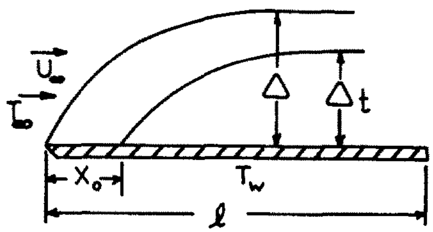


FIG. 1. Schematic diagram of the problem.

In the initial phases of the study a value of 0·1 was used for x_0^*. The effects of smaller values were also investigated.

A magnitude scaled analog computer circuit suitable for generating the solution to equations (22) and (23) is given in Fig. 2. As noted on this circuit, the relation $x^* = 0{\cdot}1\ t$, is assumed between the problem independent variable x^* and the computer independent variable t. As a consequence of this assumption, the solution time on the computer is 10 s (i.e. in 10 s x^* covers its range of 0–1).

As indicated earlier, the particular problem considered is associated with an unheated starting length given by $x_0^* \times 0{\cdot}1$. The implementation of this requires that the output of Amp 45 (Fig. 2) remain at zero for x^* in the interval $(0, x_0^*)$. This is achieved by generating a logic signal L whose sense changes form FALSE to TRUE at $x^* = x_0^*$ (integrator Amp 20 and comparator C 19 of Fig. 3). This logic signal in turn is used to control switch S1 (Fig. 3) and thereby disconnect the input to Amp 45 when $0 \leqslant x^* \leqslant x_0^*$. This then achieves the desired effect.

The purpose of the additional switch S2 in the circuit of Fig. 3 is to disable the division operation in the interval $0 \leqslant x^* \leqslant x_0^*$. The

* All computational work for this problem was carried out on the EAI–680 analog/hybrid computer facility of the Analysis Laboratory, National Research Council, Ottawa, Canada.

Table 1. Parameters for Run Number 1 to Run number 29 with $x_0^* = 0{\cdot}1$

Re_∞ No.	Pr_∞ No.	A	B	Run No.
		1·0	0·0	1
		0·7	0·0	2
	2·0	0·3	0·0	3
		0·0	0·0	−4
		−0·5	0·0	5
80 000		−0·5	0·1	6
	5·0	1·5	−0·1	7
		0·5	−0·2	8
		1·5	−0·1	9
	10·0	1·0	−0·2	10
		0·5	−0·1	11
		1·0	−0·2	12
		0·1	−0·1	13
	2·0	0·0	0·0	14
		−0·5	0·0	15
200 000		−0·5	0·2	16
		1·5	−0·2	17
	5·0	1.5	−0·1	18
		0·5	−0·1	19
		2·0	−0·2	20
	10·0	0·8	−0·2	21
		0·3	−0·1	22
	2·0	1·0	−0·2	23
320 000		0·0	0·0	24
		2·0	−0·2	25
	5·0	1·5	−0·1	26
		0·5	−0·1	27
		1·5	−0·2	28
	10·0	0·5	−0·1	29

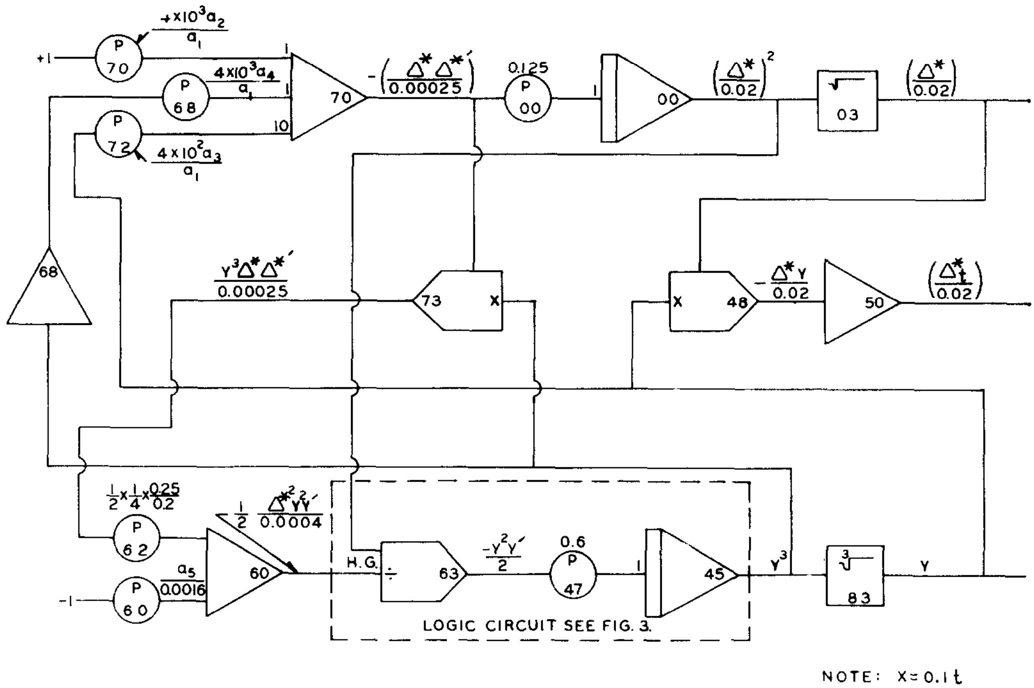


FIG. 2. Magnitude scaled analog computer circuit diagram of the problem.

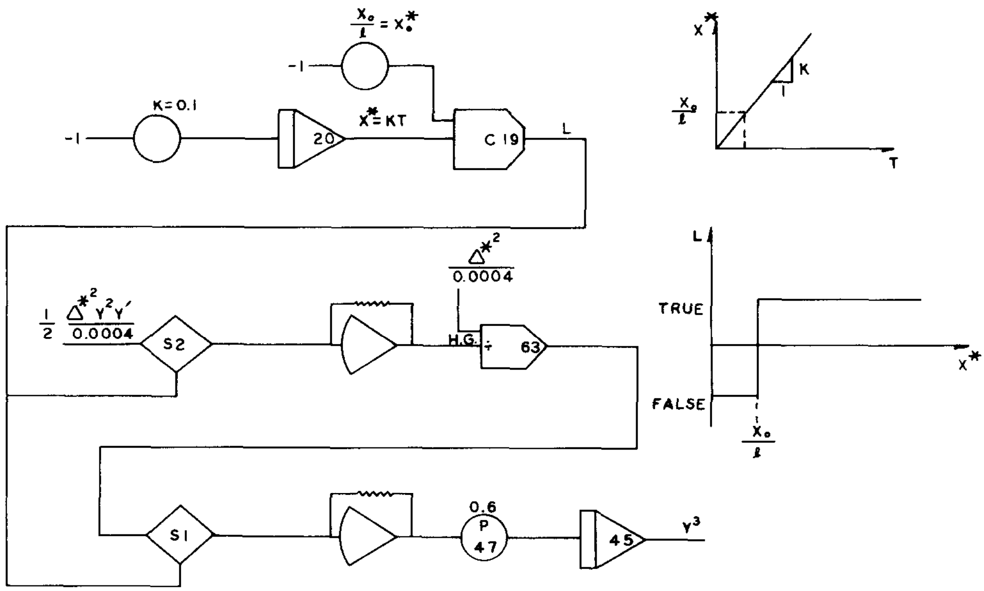


FIG. 3. The logic circuit of the problem.

disabling of this operation is necessary to avoid overloading the division unit in this interval where the variable Y is held at zero.

The ranges of the parameters Re_∞, Pr_∞, A and B used in this study are shown in Table 1. Viscosity coefficients A and conductivity coefficients B have been selected based on two criteria,

(1) physical consideration, and
(2) the value of $Y(\Delta_t/\Delta)$ remaining less than 1 over the whole solution.

In general, for incompressible fluids, the thermal conductivity is slightly dependent on temperature. In this study, conductivity is assumed slightly increasing with temperature. On the other hand, the viscosity is always rapidly decreasing with temperature.

For $x_0^* = 0{\cdot}1$, 29 runs were performed for different combination of parameters as shown in Table 1. Solutions were also obtained for values of $x_0^* < 0{\cdot}1$; in particular one run was obtained for $x_0^* = 0{\cdot}00286$, another run for $x_0^* = 0{\cdot}000572$ and two more runs for $x_0^* = 0{\cdot}01$. Table 2 provides a summary of these additional runs. For these cases the scale factor associated with $Y'\ Y^2$ had to be increased from its earlier value of 2.

Table 2. Parameters for Run Number 30 to Run Number 33 with $Re_\infty = 80\,000$

Pr_∞ No.	x_0^*	A	B	Run No.
2·0	0·01	0·0	0·0	30
	0·00286	1·0	−0·2	31
10·0	0·01	1·0	−0·2	32
	0·000572	1·0	−0·2	33

The runs with x_0^* set to 0·000572 closely approximate the case of simultaneous development of momentum and thermal boundary layers. Notice that Δ^* and Δ_t^* are direct outputs of the computer. Local Nusselt number, Nu_x, and local friction, f, were calculated according to the following relations i.e. equations (28) and (29).

From Fourier's law

$$q = -k_w\left(\frac{\partial\theta}{\partial y}\right)_{y=0}(T_\infty - T_w)$$

$$= -\frac{3k_w}{2}\frac{1}{\Delta_t}(T_\infty - T_w) \qquad (24)$$

also,

$$q = h(T_w - T_\infty). \qquad (25)$$

From equations (24) and (25), it follows

$$h = \frac{3k_w}{2}\frac{1}{\Delta_t}. \qquad (26)$$

Rearranging the terms in equation (26), one obtains

$$Nu_x = \frac{hx}{k_w} = \frac{3}{2}\frac{x}{\Delta_t} \qquad (27)$$

or

$$Nu_x = \frac{3}{2}\frac{x^*}{\Delta_t^*}. \qquad (28)$$

By definition

$$f = \frac{\mu_w(\partial u/\partial y)_{y=0}}{\rho u_\infty^2/2} = \frac{3\nu_\infty(1 + A)}{u_\infty\Delta}$$

or

$$f = \frac{3(1 + A)}{Re_\infty\Delta^*}. \qquad (29)$$

The results are presented and discussed in the next section.

RESULTS AND DISCUSSIONS

For the case of $x_0^* = 0{\cdot}1$, 29 runs were obtained with the parameters for each run shown in Table 1. Some of these results are shown from Fig. 4 to Fig. 9. These include

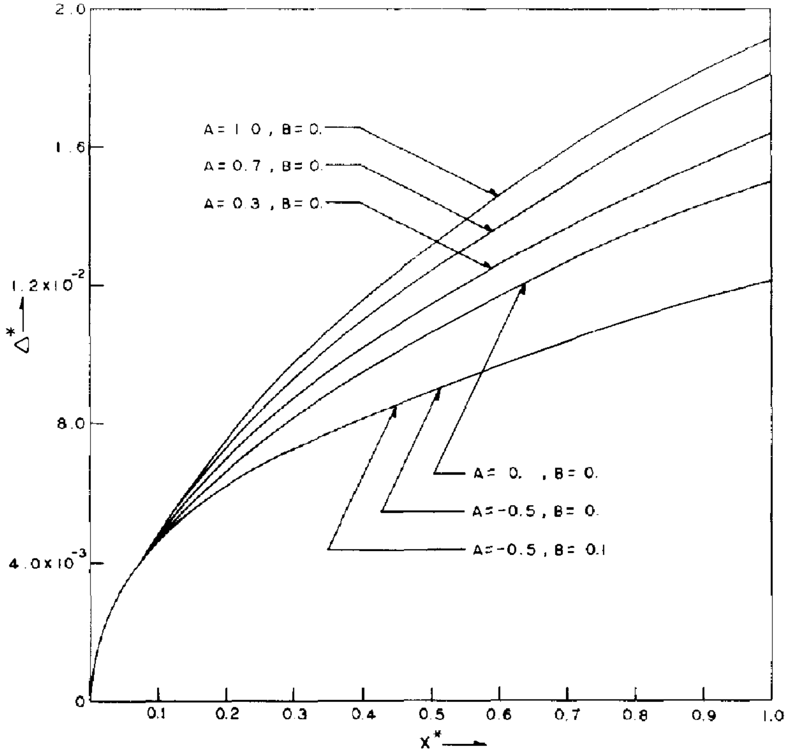


FIG. 4. Momentum boundary layer thickness for $Re_{\infty} = 80\,000$, $Pr_{\infty} = 2$.

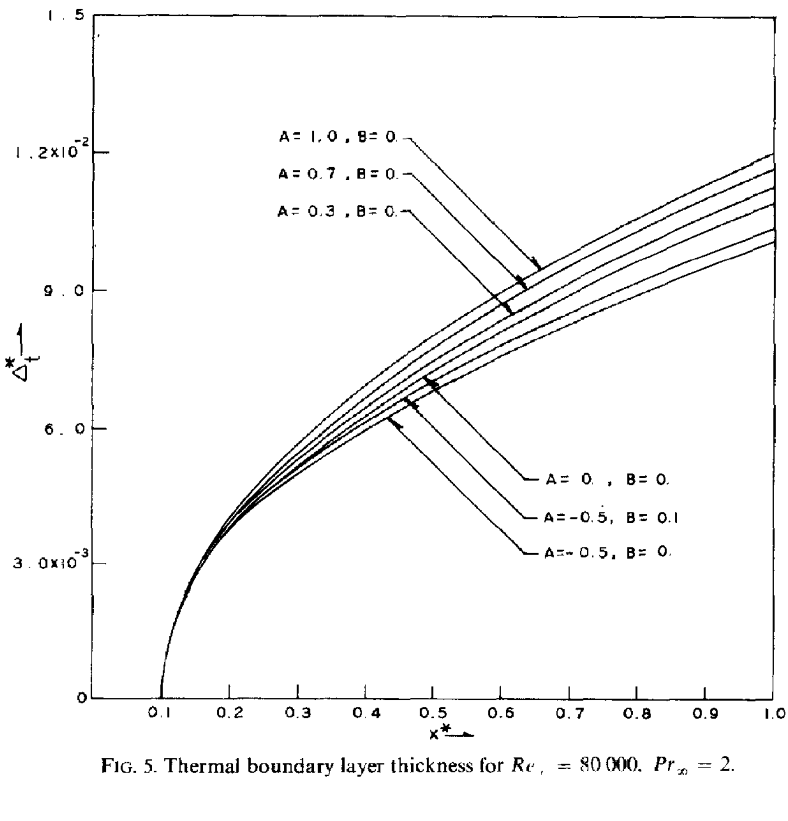


FIG. 5. Thermal boundary layer thickness for $Re_{\infty} = 80\,000$, $Pr_{\infty} = 2$.

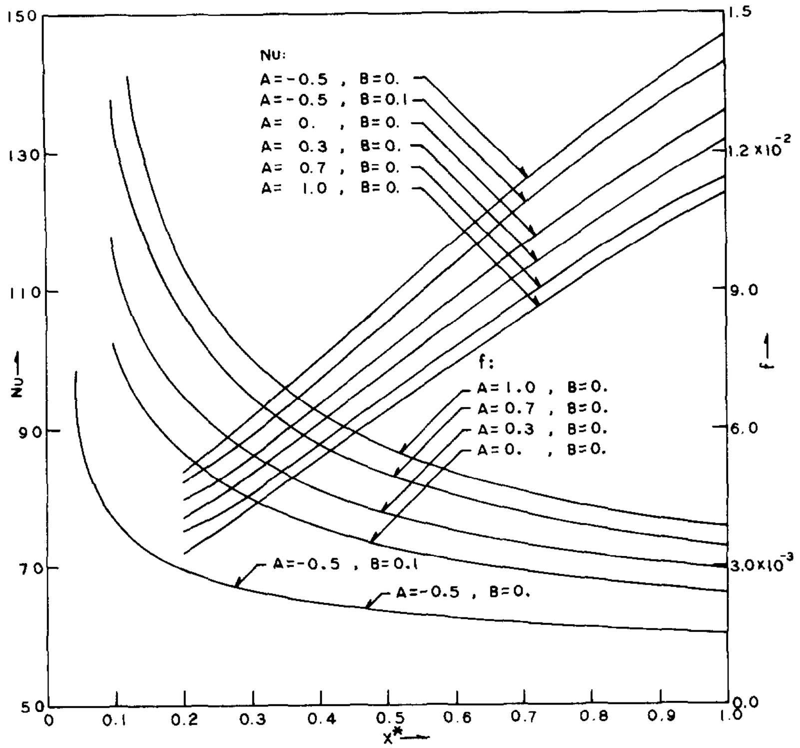


FIG. 6. Local Nusselt number and local friction factor for $Re_\infty = 80\,000$, $Pr_\infty = 2$.

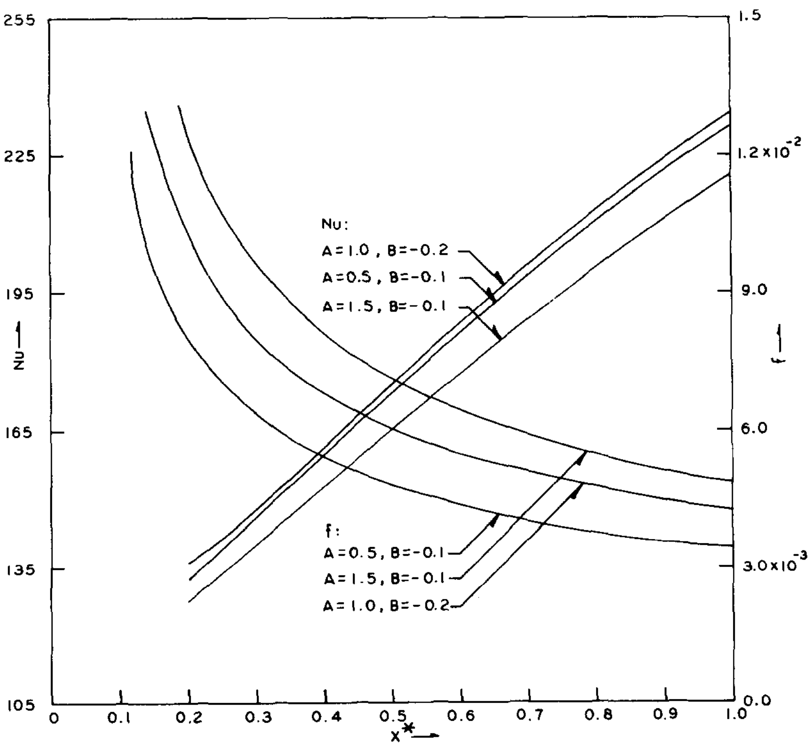


FIG. 7. Local Nusselt number and local friction factor for $Re_\infty = 80\,000$, $Pr_\infty = 10$.

momentum and thermal boundary layer thickness, local Nusselt number and local friction factor. Figures 10 and 11 show the cases of $x_0^* = 0{\cdot}01$ and $0{\cdot}00286$ at $Re_\infty = 80\,000$ and $Pr_\infty = 2$. Figures 12 and 13 show the cases of $x_0^* = 0{\cdot}01$ and $0{\cdot}000572$ at $Re_\infty = 80\,000$ and $Pr_\infty = 10$. A comparison has been made for the case of constant properties (i.e. $A = 0{\cdot}0$, $B = 0{\cdot}0$) between the results from this study and Blasius' exact solutions, the results from a typical run is shown in Fig. 14.

For constant properties case equations (22) and (23) become

$$2Y^2\Delta^{*2}Y' = \frac{128}{9Pr_\infty Re_\infty}\left(\frac{3}{5}\right) - Y^3\Delta^*\Delta^{*\prime} \quad (30)$$

$$\frac{21}{320}Re_\infty\Delta^*\Delta^{*\prime} = \frac{3}{5}. \quad (31)$$

Equation (31) can be solved as

$$\Delta = 4{\cdot}26\sqrt{(v_x/u_\infty)} \quad (32)$$

or

$$\frac{\Delta}{x} = \frac{4{\cdot}26}{\sqrt{Re_x}}. \quad (33)$$

Substituting equation (31) into equation (30), one obtains

$$\frac{15}{14}\frac{v}{\alpha}\left(Y^3 + 4\ x\ Y^2\frac{dY}{dx}\right) = 1. \quad (34)$$

The solution of equation (34) [12] is

$$Y = \frac{1}{1{\cdot}024\ \sqrt[3]{(Pr)}}\ \sqrt[3]{[1-(x_0/x)^{\frac{3}{4}}]}. \quad (35)$$

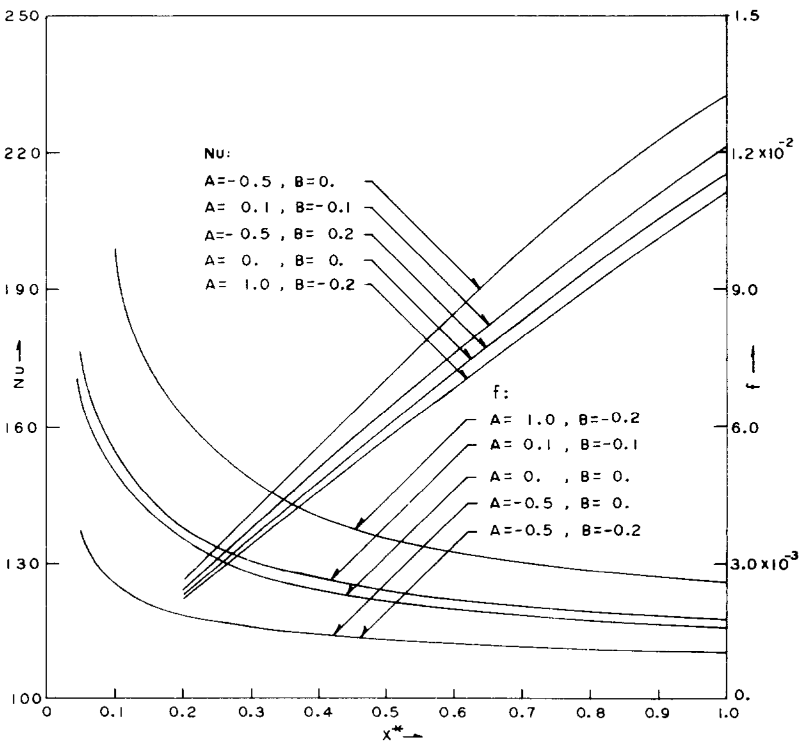


FIG. 8. Local Nusselt number and local friction factor for $Re_\infty = 200\,000$, $Pr_\infty = 2$.

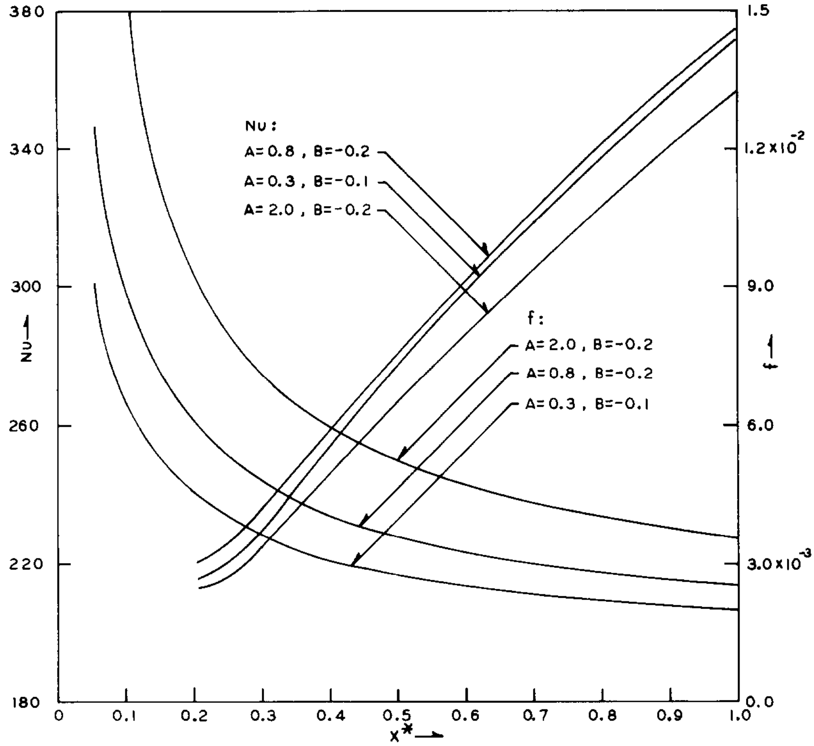


FIG. 9. Local Nusselt number and local friction factor for $Re_\infty = 200\,000$, $Pr_\infty = 10$.

Now

$$\Delta_t = \Delta Y = \frac{4{\cdot}16x}{\sqrt{(Re_x)}\,\sqrt[3]{Pr}}\,\sqrt[3]{[1-(x_0/x)^{3/4}]}\,. \quad (36)$$

Comparing equation (33) with Blasius' exact solution [12], the momentum boundary layer thickness is in error by 14·8 per cent. Because of the similarity between Y as given in equation (35) and the exact solution, it follows that the error in the thermal boundary layer thickness is also in the order of 14·8 per cent.

From equation (29)

$$f = \frac{3}{Re_\infty \Delta^*} \quad (37)$$

($A = 0{\cdot}0$, for constant properties). Substituting equation (33) into equation (37), one obtains

$$f = 0{\cdot}704\,\frac{1}{\sqrt{(Re_x)}}. \quad (38)$$

The result is in error by 6 per cent relative to the exact solution (see also Fig. 14).

Substituting equation (36) into equation (27), one obtains

$$Nu_x = 0{\cdot}36\,\sqrt[3]{(Pr)}\,\sqrt{(Re_x)}\,\frac{1}{\sqrt[3]{[1-(x_0/x)^{3/4}]}}. \quad (39)$$

The result for the Nusselt number is in error by 8·34 per cent relative to the exact solution (see also Fig. 14).

For the case of flow over a flat plate with variable properties, there are no results available from the literature. Consequently, only some

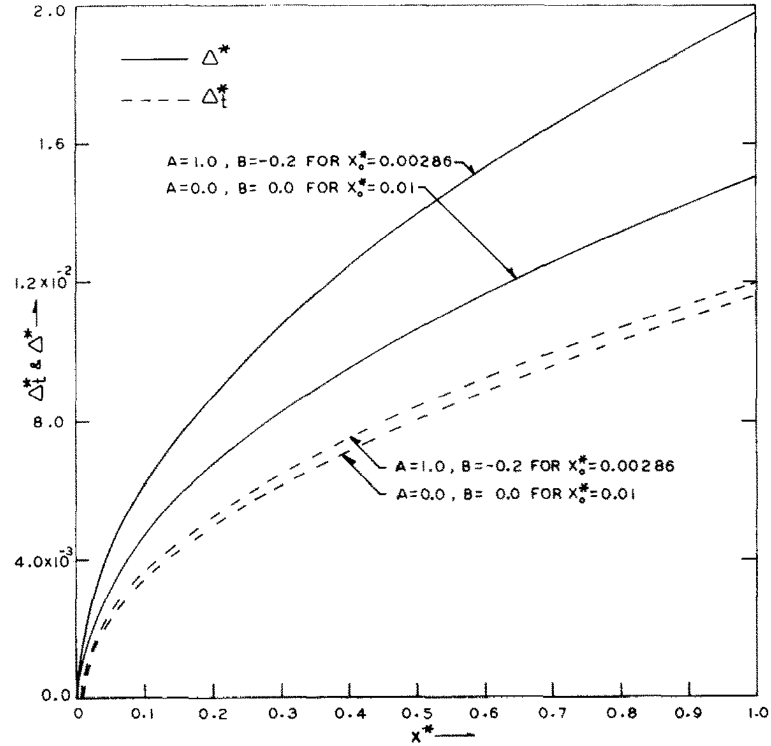


FIG. 10. Momentum and thermal boundary layer thickness for $Re_\infty = 80\,000$. $Pr_\infty = 2$ when x_0^* approaching zero.

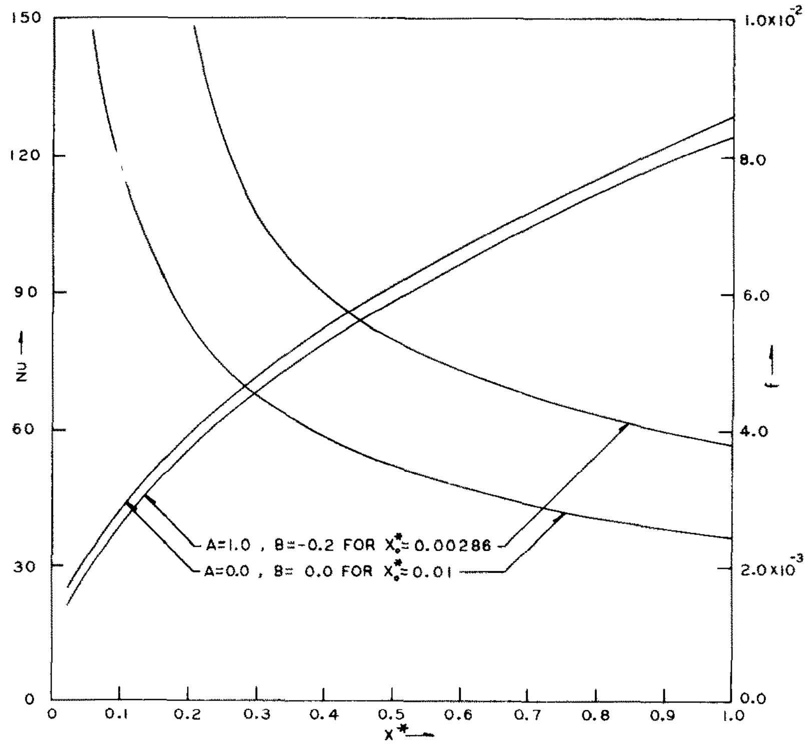


FIG. 11. Local Nusselt number and local friction factor for $Re_\infty = 80\,000$. $Pr_\infty = 2$ when x_0^* approaching zero.

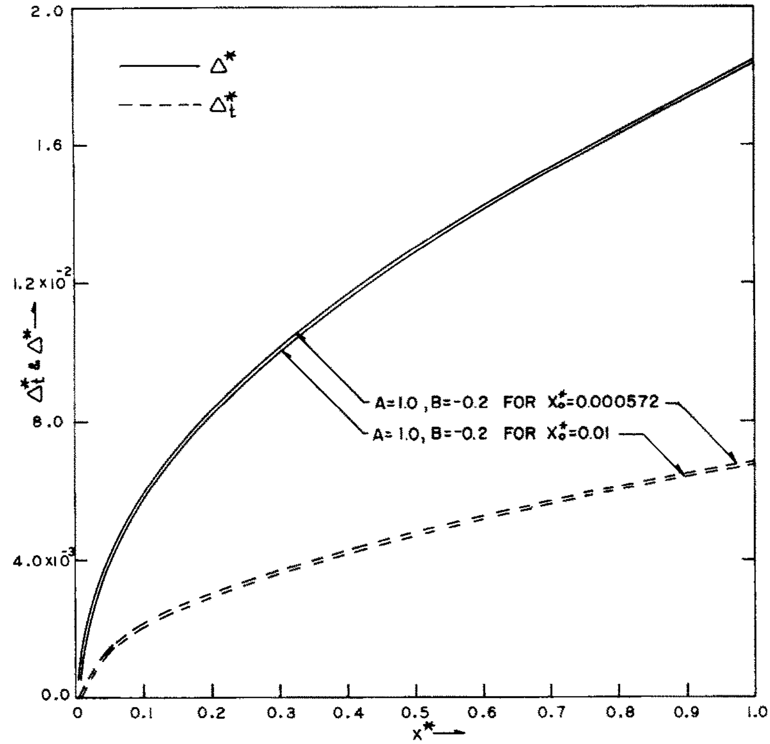


FIG. 12. Momentum and thermal boundary layer thickness for $Re_\infty = 80\,000$, $Pr_\infty = 10$ when x_0^* approaching zero.

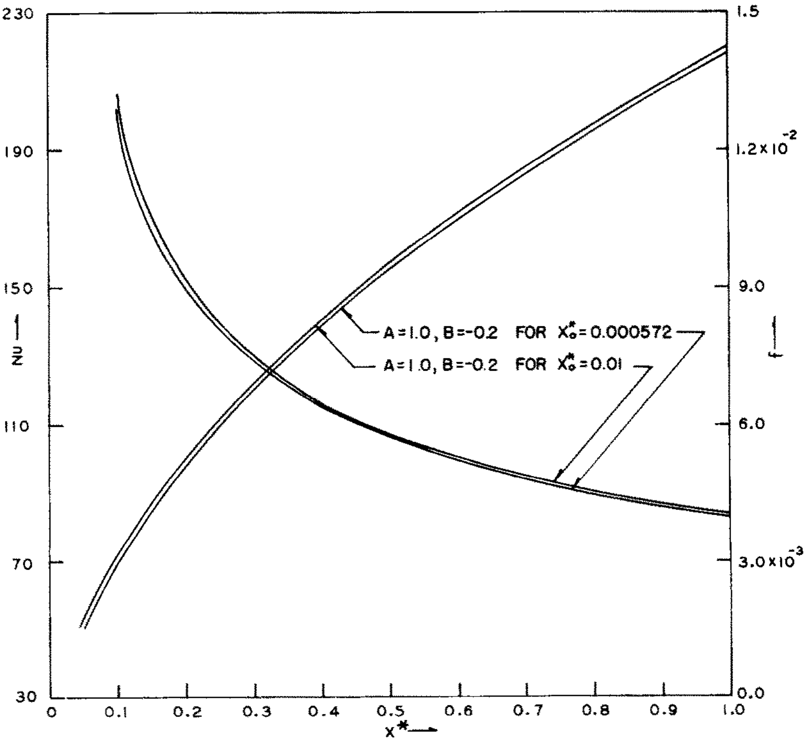


FIG. 13. Local Nusselt number and local friction factor for $Re_\infty = 80\,000$, $Pr_\infty = 10$ when x_0^* approaching zero.

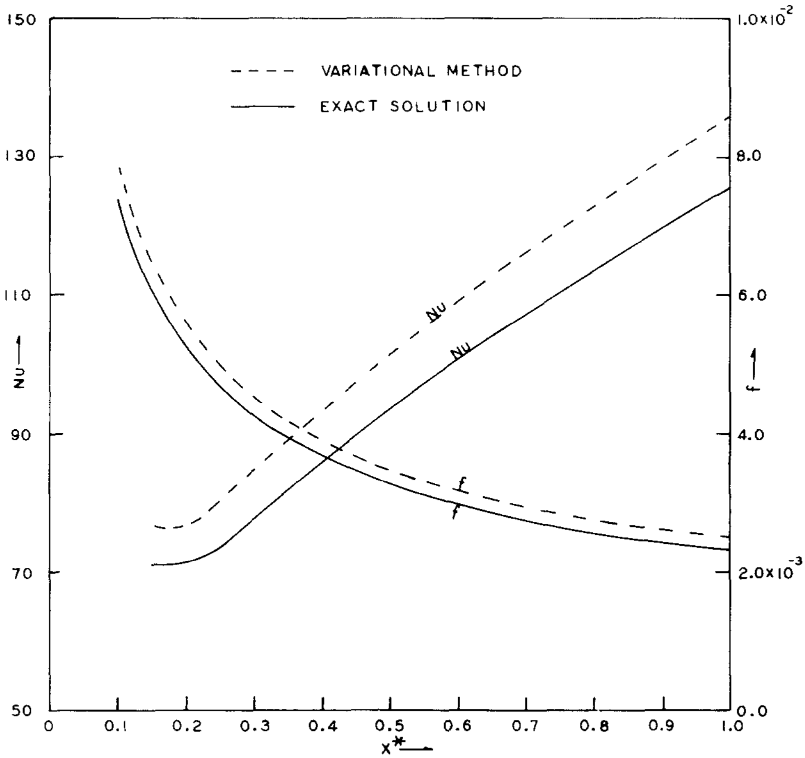


FIG. 14. Comparison of Nusselt number and friction factor for constant properties case at $Re_{\infty} = 80\,000$, $Pr_{\infty} = 2$.

qualitative remarks can be made regarding the effects of the parameters on the flow properties. Some of these qualitatively agree with results from the literature [13] for the flow through a pipe with variable properties.

(1) For constant conductivity coefficients B, Nusselt number increases with decreasing viscosity coefficients A for all x^*. But friction factor increases with increasing A for all x^*. This is due to the fact that positive A indicates cooling of liquid in the flow, thus corresponding increase in viscosity near the wall slows down the flow, results in a lower rate of heat transfer, relative to the constant properties case.

(2) The curves in Fig. 6 and Fig. 8 show that for constant viscosity coefficients A, Nusselt number decreases with increasing B (conductivity coefficient) for all x^*, while friction factor remains unchanged.

(3) For constant Reynolds number, Nusselt number increases with Prandtl number for all x^*.

(4) The curves in Fig. 13 show that for constant Reynolds number and constant Prandtl number, Nusselt number decreases with decreasing unheated starting length x_0^* for all x^*, while friction factor decreases slightly with decreasing unheated starting length x_0^* for all x^*.

It is believed that the results from this study can be improved by assuming a higher order velocity profile and a higher order temperature profile in the calculation process. Similarly, more realistic expressions of viscosity and thermal conductivity would be useful.

Nevertheless, the variational method combined with an analog/hybrid computer solution technique has been found to be very fruitful in the solution of this complex problem. It is felt that the great potential of such a combined technique still remains to be fully exploited in other engineering problems.

ACKNOWLEDGEMENTS

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REFERENCES

1. P. GLANSDORFF, I. PRIGOGINE and D. F. HAYS, Variational properties of a viscous liquid at a non-uniform temperature, *Physics Fluids* **5** (2), 144–149 (1962).
2. D. F. HAYS, A variational formulation applied to Couette and Poiseuille flow, *Bul. Acad. Belg. Classe Sciences* **49**, 576–602 (1963).
3. H. W. BUTLER and R. L. RACKLEY, Application of a variational formulation to non-equilibrium fluid flow, *Int. J. Heat Mass Transfer* **10**, 1255–1266 (1967).
4. P. GLANSDORFF and I. PRIGOGINE, On a general evolution criterion in macroscopic physics, *Physica* **30**, 351–374 (1964).
5. D. F. HAYS, An extended variational method applied to Poiseuille flow: temperature dependent viscosity, *Int. J. Heat Mass Transfer* **9**, 165–170 (1966).
6. R. J. DONNELLY, R. HERMAN and I. PRIGOGINE, Symposium on *Non-equilibrium Thermodynamics: Variational Techniques and Stability*, Part 1. University of Chicago Press (1966).
7. R. S. SCHECHTER, *The Variational Method in Engineering*. Chapter 5. McGraw–Hill (1967).
8. H. SCHLICTING, *Boundary Layer Theory*, Chapters VII and X, Sixth Edition. McGraw–Hill (1968).
9. W. M. KAYS, *Convective Heat and Mass Transfer*. Chapter 4. McGraw–Hill (1966).
10. R. S. SCHECHTER, *The Variational Method in Engineering*, Chapter 1. McGraw–Hill (1967).
11. F. B. HILDERBRAND, *Method of Applied Mathematics*. Chapter 2. Prentice–Hall (1952).
12. E. R. G. ECKERT and R. M. DRAKE, JR., *Heat and Mass Transfer*. Chapter 7. McGraw–Hill (1959).
13. K. T. YANG, Laminar forced convection of liquids in tubes with variable viscosity, *J. Heat Transfer* **84C**, 353–362 (1962).

APPENDIX

Derivation of the Coupled Equations from the Variational Formulation of the Problem

First, we re-write equation (13)

$$E = \iint \left[\frac{k^\circ}{2}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{\mu^\circ}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \rho C_p^\circ u^\circ T\frac{\partial T^\circ}{\partial x} + \rho C_p^\circ v^\circ T\frac{\partial T^\circ}{\partial y} - \rho u^{\circ 2}\frac{\partial u}{\partial x} - \rho u^\circ v^\circ \frac{\partial u}{\partial y}\right]\mathrm{d}x\mathrm{d}y$$

$$+ \int_0^{\Delta}(\rho u^{\circ 2} u|_{x=l} - \rho u_\infty^3)\,\mathrm{d}y - \int_0^{l}\rho u_\infty^2 v^\circ|_{y=\Delta}\mathrm{d}x. \qquad (A.1)$$

Since

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3$$

it follows that

$$u = u_\infty\left[\frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3\right]$$

$$\frac{\partial u}{\partial x} = u_\infty\left[-\frac{3}{2}\frac{y\Delta'}{\Delta^2} + \frac{3}{2}\frac{y^3\Delta'}{\Delta^4}\right]$$

and

$$\frac{\partial u}{\partial y} = u_\infty\left[\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2\Delta}\frac{y^2}{\Delta^2}\right].$$

From the continuity equation

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

therefore

$$v = \int_0^y -\frac{\partial u}{\partial x}\mathrm{d}y$$

$$= u_\infty\int_0^y\left[\frac{3}{2}\left(\frac{y\Delta'}{\Delta^2}\right) - \frac{3}{2}\left(\frac{y^3\Delta'}{\Delta^4}\right)\right]\mathrm{d}y$$

$$= u_\infty\left[\frac{3}{4}\left(\frac{y^2\Delta'}{\Delta^2}\right) - \frac{3}{8}\left(\frac{y^4\Delta'}{\Delta^4}\right)\right].$$

Also, it is assumed that

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\left(\frac{y}{\Delta_t}\right) - \frac{1}{2}\left(\frac{y}{\Delta_t}\right)^3$$

and so

$$T = (T_\infty - T_w)\left[\frac{3}{2}\left(\frac{y}{\Delta_t}\right) - \frac{1}{2}\left(\frac{y}{\Delta_t}\right)^3\right] + T_w$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w)\left[\frac{3}{2\Delta_t} - \frac{3}{2\Delta_t}\left(\frac{y}{\Delta_t}\right)^2\right]$$

and

$$\frac{\partial T}{\partial x} = (T_\infty - T_w)\left[-\frac{3}{2}\left(\frac{y\Delta_t'}{\Delta_t^2}\right) + \frac{3}{2}\left(\frac{y^3\Delta_t'}{\Delta_t^4}\right)\right]$$

Also

$$\mu = \mu_\infty(1 + A\theta)$$

and

$$k = k_\infty(1 + B\theta).$$

Substituting all the above expressions into equation (A.1) gives

$$E = \iint \Bigg\{ \frac{k_\infty - Bk_\infty\left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^3} - 1\right)}{2}(T_\infty - T_w)^2 \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3}\right)^2 + \frac{\mu_\infty - A\mu_\infty\left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^{\circ 3}} - 1\right)}{2}$$

$$+ (u_\infty)^2\left(\frac{3}{2\Delta} - \frac{3y^2}{2\Delta^3}\right)^2 + \rho C_p u_\infty\left(\frac{3y}{2\Delta^\circ} - \frac{y^3}{2\Delta^{\circ 3}}\right)$$

$$\left[(T_\infty - T_w)\left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^3}\right) + T_w\right](T_\infty - T_w)$$

$$\left(\frac{-3y\Delta_t^{\circ\prime}}{2\Delta_t^{\circ 2}} + \frac{3y^3\Delta_t^{\circ\prime}}{2\Delta_t^{\circ 4}}\right) + \rho C_p^\circ u_\infty\left(\frac{3y^2\Delta^{\circ\prime}}{4\Delta^{\circ 2}} - \frac{3}{8}\frac{y^4\Delta^{\circ\prime}}{\Delta^{\circ 4}}\right)$$

$$\left[(T_\infty - T_w)\left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^3}\right) + T_w\right](T_\infty - T_w)$$

$$\left(\frac{3}{2\Delta_t^\circ} - \frac{3y^2}{2\Delta_t^{\circ 3}}\right) - \rho u_\infty^3\left(\frac{3}{2}\frac{y}{\Delta^\circ} - \frac{y^3}{2\Delta^{\circ 3}}\right)^2$$

$$\left(-\frac{3y\Delta'}{2\Delta^2} + \frac{3}{2}\frac{y^3\Delta'}{\Delta^4}\right) - \rho u_\infty^3\left(\frac{3}{2}\frac{y}{\Delta^\circ} - \frac{y^3}{2\Delta^{\circ 3}}\right)$$

$$\left(\frac{3y^2\Delta^{\circ\prime}}{4\Delta^{\circ 2}} - \frac{3}{8}\frac{y^4\Delta^{\circ\prime}}{\Delta^{\circ 4}}\right)\left(\frac{3}{2\Delta} - \frac{3y^2}{2\Delta^3}\right)\Bigg\}\,\mathrm{d}x\mathrm{d}y$$

$$+ \int_0^\Delta (\rho u^{\circ 2} u|_{x=l} - \rho u_\infty^3)\,\mathrm{d}y - \int_0^l \rho u_\infty^2 v^\circ|_{y=\Delta}\,\mathrm{d}x. \tag{A.2}$$

Taking the variation of E (equation (A.2)) with respect to Δ_t, we obtain

$$\delta E = \int_0^l \int_0^{\Delta t} \Bigg\{\left[k_\infty - Bk_\infty\left(\frac{3}{2}\frac{y}{\Delta_t^\circ} - \frac{y^3}{2\Delta_t^{\circ 3}} - 1\right)\right](T_\infty - T_w)^2$$

$$\left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3}\right)\left(\frac{-3}{2\Delta_t^2} + \frac{9y^2}{2\Delta_t^4}\right) + \rho C_p^\circ u_\infty\left(\frac{3}{2}\frac{y}{\Delta} - \frac{y^3}{2\Delta^{\circ 3}}\right)$$

$$(T_\infty - T_w)^2\left(\frac{-3y}{2\Delta_t^2} + \frac{3y^3}{2\Delta_t^4}\right)\left(-\frac{3y\Delta_t^{\circ\prime}}{2\Delta_t^{\circ 2}} + \frac{3y^3\Delta_t^{\circ\prime}}{2\Delta_t^{\circ 4}}\right)$$

$$+ \rho C_p^\circ u_\infty\left(\frac{3y^2\Delta^{\circ\prime}}{4\Delta^{\circ 2}} - \frac{3}{8}\frac{y^4\Delta^{\circ\prime}}{\Delta^{\circ 4}}\right)(T_\infty - T_w)^2\left(\frac{-3y}{2\Delta_t^2} + \frac{3y^3}{2\Delta_t^4}\right)$$

$$\left(\frac{3}{2\Delta_t^\circ} - \frac{3y^2}{2\Delta_t^{\circ 3}}\right)\Bigg\}\,\delta\Delta_t\,\mathrm{d}x\mathrm{d}y. \tag{A.3}$$

There is no longer any need to distinguish between the varied (Δ_t) and unvaried (Δ_t°) versions of thermal boundary layer thickness. Dropping the superscript and integrating y from 0 to Δ_t, we find

$$\delta E = \int_0^l \Bigg[-\frac{3}{5}(T_\infty - T_w)^2\frac{k_\infty}{\Delta_t^2} - \frac{177}{320}B(T_\infty - T_w)^2\frac{k_\infty}{\Delta_t^2}$$

$$+ \rho C_p u_\infty (T_\infty - T_w)^2 \frac{1}{\Delta_t^2}\left(\frac{9}{64}\frac{\Delta_t^2\Delta_t'}{\Delta} - \frac{3}{160}\frac{\Delta_t^4\Delta_t'}{\Delta^3}\right.$$

$$\left.-\frac{9}{128}\frac{\Delta_t^3\Delta'}{\Delta^2} + \frac{9}{640}\frac{\Delta_t^5\Delta'}{\Delta^4}\right)\Bigg]\delta\Delta_t\,\mathrm{d}x. \tag{A.4}$$

Since δE must vanish for all $\delta\Delta_t$, therefore

$$-\frac{3}{5}\frac{k_\infty}{\Delta_t^2} - \frac{177}{320}B\frac{k_\infty}{\Delta_t^2} + \frac{\rho C_p u_\infty}{\Delta_t^2}\left(\frac{9}{64}\frac{\Delta_t^2\Delta_t'}{\Delta} - \frac{3}{160}\frac{\Delta_t^4\Delta_t'}{\Delta^3} - \frac{9}{128}\frac{\Delta_t^3\Delta'}{\Delta^2} + \frac{9}{640}\frac{\Delta_t^5\Delta'}{\Delta^4}\right) = 0 \tag{A.5}$$

Introducing the ratio $Y = \dfrac{\Delta_t}{\Delta}$, equation (A.5) becomes

$$-\frac{3}{5}\alpha - \frac{177}{320}\alpha B + u_\infty\left[\frac{9}{128}(Y^3\Delta\Delta' + 2Y^2\Delta^2 Y') - \frac{3}{640}(4Y^4\Delta^2 Y' + Y^5\Delta\Delta')\right] = 0. \tag{A.6}$$

Similarly, we take the variation of E (equation (A.2)) with respect to Δ,

$$\delta E = \int_0^l\int_0^\Delta \Bigg\{\left[\mu_\infty - A\mu_\infty\left(\frac{3}{2}\frac{y}{\Delta_t^\circ} - \frac{y^3}{2\Delta_t^{\circ 3}} - 1\right)\right]$$

$$(u_\infty)^2\left(\frac{3}{2\Delta} - \frac{3y^2}{2\Delta^3}\right)\left(-\frac{3}{2\Delta^2} + \frac{9y^2}{2\Delta^4}\right)\delta\Delta$$

$$-\rho u_\infty^3\left(\frac{3y}{2\Delta^\circ} - \frac{y^3}{2\Delta^{\circ 3}}\right)\left(\frac{3y^2\Delta^{\circ\prime}}{4\Delta^{\circ 2}} - \frac{3y^4\Delta^{\circ\prime}}{8\Delta^{\circ 4}}\right)$$

$$\left(-\frac{3}{2\Delta^2} + \frac{9y^2}{2\Delta^4}\right)\delta\Delta - \rho u_\infty^3\left(\frac{3y}{2\Delta^\circ} - \frac{y^3}{2\Delta^{\circ 3}}\right)^2$$

$$\left[\left(\frac{3y\Delta'}{\Delta^3} - \frac{6y^3\Delta'}{\Delta^5}\right)\delta\Delta + \left(-\frac{3}{2}\frac{y}{\Delta^2} + \frac{3}{2}\frac{y^3}{\Delta^4}\right)\delta\Delta'\right]\Bigg\}\,\mathrm{d}x\mathrm{d}y$$

$$+\int_0^\Delta\left[\rho u_\infty^3\left(\frac{3y}{2\Delta^\circ} - \frac{1}{2}\frac{y^3}{\Delta^{\circ 3}}\right)^2\left(-\frac{3y}{2\Delta^2} + \frac{3y^3}{2\Delta^4}\right)\delta\Delta\right]_{x=l}\mathrm{d}y. \tag{A.7}$$

As before, there is no longer any need to distinguish between the varied (Δ) and unvaried (Δ°) versions of the momentum boundary layer thickness. By using the fact that $\delta\Delta' = \mathrm{d}(\delta\Delta)/\mathrm{d}x$, the term involving $\delta\Delta'$ on the right-hand side of equation (A.7) can be written as

$$I = -\int_0^l\int_0^\Delta \rho u_\infty^3\left(\frac{3}{2}\frac{y}{\Delta} - \frac{y^3}{2\Delta^3}\right)^2 \left(-\frac{3y}{2\Delta^2} + \frac{3y^3}{2\Delta}\right)_4 \delta\Delta' \mathrm{d}x\mathrm{d}y$$

$$= -\int_0^l\int_0^\Delta \frac{3}{8}\rho u_\infty^3\left(\frac{3y}{\Delta} - \frac{y^3}{\Delta}\right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4}\right) \frac{\mathrm{d}(\delta\Delta)}{\mathrm{d}x}\mathrm{d}x\mathrm{d}y. \quad (A.8)$$

Integration by parts yields.

$$I = -\int_0^\Delta \frac{3}{8}\rho u_\infty^3\left(\frac{3y}{\Delta} - \frac{y^3}{\Delta^3}\right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4}\right)\delta\Delta|_{x=l}\mathrm{d}y$$

$$+\int_0^l\int_0^\Delta \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3}{8}\rho u_\infty^3\left(\frac{3y}{\Delta} - \frac{y^3}{\Delta^3}\right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4}\right)\right]\delta\Delta\mathrm{d}x\mathrm{d}y. \quad (A.9)$$

Finally, integrating y from 0 to Δ of equation (A.7), we obtain

$$\delta E = \int_0^l \left[-\frac{3}{5}\mu_\infty u_\infty^2\frac{1}{\Delta^2} - \frac{9}{4}A\mu_\infty u_\infty^2\left(\frac{3}{8}\frac{\Delta_t}{\Delta^3} - \frac{\Delta_t^3}{6\Delta^5} + \frac{3}{80}\frac{\Delta_t^5}{\Delta^7}\right) \frac{21}{320}\rho u_\infty^3\frac{\Delta'}{\Delta}\right]\delta\Delta\mathrm{d}x. \quad (A.10)$$

Since δE must vanish for all $\delta\Delta$, therefore

$$-\frac{3}{5}\mu_\infty u_\infty^2\frac{1}{\Delta^2} - \frac{9}{4}A\mu_\infty u_\infty^2\left(\frac{3}{8}\frac{\Delta_t}{\Delta^3} - \frac{1}{6}\frac{\Delta_t^3}{\Delta^5} + \frac{3}{80}\frac{\Delta_t^5}{\Delta^7}\right) + \frac{21}{320}\rho u_\infty^3\frac{\Delta'}{\Delta} = 0. \quad (A.11)$$

Introducing the ratio $Y = \frac{\Delta_t}{\Delta}$, equation (A.11) becomes

$$-\frac{3}{5}\mu_\infty - \frac{9}{4}A\mu_\infty\left(\frac{3}{8}Y - \frac{1}{6}Y^3 + \frac{3}{80}Y^5\right) + \frac{21}{320}\rho u_\infty\Delta\Delta' = 0. \quad (A.12)$$

Since Δ_t is assumed to be smaller than Δ, it follows that $Y < 1$, hence equations (A.6) and (A.12) can be simplified by neglecting the higher order terms Y^4 and Y^5.

$$-\frac{3}{5}\alpha - \frac{177}{320}\alpha B + u_\infty\left[\frac{9}{128}(Y^3\Delta\Delta' + 2Y^2\Delta^2Y')\right] = 0 \quad (A.13)$$

$$-\frac{3}{5}\mu_\infty - \frac{9}{4}A\mu_\infty\left(\frac{3}{8}Y - \frac{1}{6}Y^3\right) + \frac{21}{320}\rho u_\infty\Delta\Delta' = 0. \quad (A.14)$$

Introducing the dimensionless quantities

$$x^* = \frac{x}{l}$$

$$\Delta^* = \frac{\Delta}{l}$$

$$\Delta_t^* = \frac{\Delta_t}{l}$$

we obtain

$$-\frac{3}{5} - \frac{177}{320}B + \frac{9Pr_\infty Re_\infty}{128}(Y^3\Delta^*\Delta^{*\prime} + 2Y^2\Delta^{*2}Y') = 0 \quad (A.15)$$

$$-\frac{3}{5} - \frac{9}{4}A\left(\frac{3}{8}Y - \frac{1}{6}Y^3\right) + \frac{21}{320}Re_\infty\Delta^*\Delta^{*\prime} = 0 \quad (A.16)$$

Finally we rewrite equation (A.15) and equation (A.16) as

$$2Y^2\Delta^{*2}Y' = a_5 - Y^3\Delta^*\Delta^{*\prime} \quad (A.17)$$

$$-a_1\Delta^*\Delta^{*\prime} = -a_2 - a_3Y + a_4Y^3 \quad (A.18)$$

where

$$a = \frac{21}{320}Re_\infty$$

$$a_2 = \frac{3}{5}$$

$$a_3 = \frac{9}{4}A\frac{3}{8}$$

$$a_4 = \frac{9}{4}A\frac{1}{6}$$

$$a_5 = \frac{128}{9Pr_\infty Re_\infty}\left(\frac{3}{5} + \frac{177}{320}B\right).$$

Equations (A.17) and (A.18) are same as equations (22) and (23).

APPLICATION D'UNE METHODE VARIATIONNELLE A L'ECOULEMENT SUR UNE PLAQUE PLANE DANS LA REGION D'ENTREE AVEC DES PROPRIETES PHYSIQUES VARIABLES

Résumé—On utilise une méthode variationnelle afin de déterminer l'écoulement sur une plaque plane dans la région d'entrée à température pariétale constante. Les propriétés physiques, par exemple la conductivité thermique et la viscosité, sont supposées être dans cette étude des fonctions linéaires de la température. Deux équations couplées ont été dérivées de la formulation variationnelle et ensuite résolues sur un calculateur hybride analogique digital. En conséquence, on donne l'épaisseur de quantité de mouvement, l'épaisseur de la couche limite thermique, le nombre de Nusselt local, et le facteur de frottement local. Dans le cas de propriétés constantes on fait une comparaison entre la solution exacte et les résultats obtenus par utilisation de la méthode d'approche suggérée dans cet article.

ANWENDUNG EINES VARIATIONSVERFAHRENS AUF DIE STRÖMUNG ÜBER EINER EBENEN PLATTE IM ANLAUFBEREICH MIT VERÄNDERLICHEN PHYSIKALISCHEN STOFFWERTEN

Zusammenfassung—Ein Variationsverfahren wurde angewendet, um die Strömung über einer ebenen Platte im Anlaufbereich bei konstanter Wandtemperatur zu bestimmen. Bei dieser Untersuchung wurden die physikalischen Stoffwerte, z.B. die Wärmeleitfähigkeit und die Viskosität, als lineare Funktion der Temperatur angenommen. Aus der Variationsformulierung wurden zwei gekoppelte Gleichungen abgeleitet und mit einem Analog-Hybrid-Rechner gelöst. Daraus werden für die Strömung die Dicken der Strömungs- und Temperatur-Grenzschicht, die örtliche Nusseltzahl und der örtliche Reibungsfaktor gefunden. Für den Fall konstanter Stoffwerte wurde zwischen den Ergebnissen der exakten Lösung und der in dieser Arbeit vorgeschlagenen Näherungslösung ein Vergleich angestellt.

ПРИМЕНЕНИЕ ВАРИАЦИОННОГО МЕТОДА К ЗАДАЧЕ О ТЕЧЕНИИ ВДОЛЬ ПЛОСКОЙ ПЛАСТИНЫ ВО ВХОДНОЙ ОБЛАСТИ С ПЕРЕМЕННЫМИ ФИЗИЧЕСКИМИ СВОЙСТВАМИ

Аннотация—Вариационный метод использовался для решения задачи о течении по плоской пластине во входной области при постоянной температуре поверхности. Физические свойства, т.е. теплопроводность и вязкость, в данном исследовании принимались линейными функциями температуры. Из вариационной формулировки выведены два связанных уравнения, которые затем были решены на аналогово-гибридной вычислительной машине. В результате для рассматриваемого течения были определены толщина пограничного слоя, локальное число Нуссельта и локальный коэффициент трения. Для случая постоянных свойств проведено сравнение точного решения с результатами, полученными с помощью методики, предложенной в данной статье.